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# Projective Coordinates and Projective Space Limit

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## Abstract

The “projective lightcone limit” has been proposed as an alternative holographic dual of an AdS space. It is a new type of group contraction for a coset  $G/H$  preserving the isometry group  $G$  but changing  $H$ . In contrast to the usual group contraction, which changes  $G$  preserving the spacetime dimension, it reduces the dimensions of the spacetime on which  $G$  is realized. The obtained space is a projective space on which the isometry is realized as a linear fractional transformation. We generalize and apply this limiting procedure to the “Hopf reduction” and obtain  $(n-1)$ -dimensional complex projective space from  $(2n-1)$ -dimensional sphere preserving  $SU(n)$  symmetry.

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# 1 Introduction

In the AdS/CFT correspondence the global symmetry is one of the most fundamental guiding principles. The global  $SO(D, 2)$  symmetry is realized in terms of not only the  $(D + 1)$ -dimensional AdS space coordinates but also the  $D$ -dimensional conformally flat space coordinates. In the usual holography this flat  $D$ -dimensional space is located at the boundary of the AdS space [1]. Instead an alternative holography was proposed [2] in which the flat  $D$ -dimensional space is replaced by a lightcone space obtained by zero-radius limit of the AdS space and the global symmetry is realized by the linear fractional transformations of the projective coordinates [3]. Under the “projective lightcone limit” the  $(D + 1)$ -dimensional AdS metric reduces into the  $D$ -dimensional conformally flat metric, while the AdS metric diverges under the boundary limit in the usual holography. The CFT on the projective lightcone is expected to be newly dual to the CFT on the usual flat space at the boundary.

The projective lightcone limit is different from the Inönü-Wigner (IW) type group contraction which does not change the number of generators, and so the number of coordinates, but changes the group structure. The projective lightcone (plc) limit changes the number of coordinates preserving the group holographically. The contraction parameter of the plc limit is the AdS radius  $R$  and the limit  $R \rightarrow 0$  gives a lightcone space. In the limit the absence of constant scale allows to use projective coordinates reducing the number of coordinates. From the view point of a coset,  $G/H$ , this limit preserves  $G$  but it is a group contraction of  $H$ . The limit is related to  $H$ -covariant quantities rather than  $G$ -covariant quantities; for a coset element  $z \rightarrow gzh$  with  $g \in G$  and  $h \in H$  the limiting parameter rescales  $z$  from the right rather than the left.

It was shown that the projective lightcone limit of the supersymmetric  $AdS_5 \times S^5$  has a possibility to construct the  $N=4$  SYM theory on the projective superspace [3]. In order to describe the  $N$  extended supersymmetric theories  $SU(N)$  internal coordinates are necessary. The harmonic superspace includes the homogeneous coordinates for the  $SU(N)$  symmetry and harmonic analysis of the  $N=2, 3$  harmonic superspaces has been well performed [4]. On the other hand the projective superspace [5] includes the projective coordinates for  $SU(N)$  and complex analysis is performed. Originally the projective coordinates are used in the Kähler potential for constructing the non-singular metric of a manifold and supersymmetric extension is obtained by replacing the projective coordinates by chiral superfields [6]. The  $N=2$  projective superspace is also useful to explore new hyperkähler metrics and related works are in [7].

In this paper we generalize the projective lightcone limit to a complex projective space limit where a limiting parameter is introduced besides the AdS radius. We examine a coset  $G/H$  with  $G=SU(n)$  case: We begin with a coordinate system for a  $(2n-1)$ -dimensional sphere with the subgroup of the coset  $H=SU(n-1)$ , and perform the limit into the  $(n-1)$ -dimensional complex projective space where the subgroup becomes  $H=SU(n-1) \otimes U(1)$ . This limiting procedure from  $S^{2n-1}$  to  $CP^{n-1}$  corresponds to the “Hopf reduction” [8] which has been studied widely [9] relating to T-duality in [10], to noncompact spaces in [11] and to the noncommutative spaces in [12].

## 2 Generalization of projective lightcone limit

### 2.1 Projective lightcone limit

In this section we review the projective lightcone (plc) limit clarifying local gauge invariance and reinterpret it from the group contraction point of view for a coset. The plc limit was introduced in [2] as follows: The  $D$ -dimensional AdS space is described by a hypersurface in terms of  $(D + 1)$ -dimensional Minkowski coordinates  $x_\mu$  as

$$\sum_{\mu=1,\dots,D,D+1} x_\mu^2 + R^2 = 0 \quad . \quad (2.1)$$

It is rewritten by projective coordinates  $X_i = x_i/x_+$  with  $i = 1, 2, \dots, D - 1$  and  $U = 1/x_+$  where  $x_\pm$  are lightcone variables. The metric of the  $D$ -dimensional AdS space is

$$ds^2 = \sum_{i=1,\dots,D-1} dx_i^2 + dx_+ dx_- = \sum_{i=1,\dots,D-1} \frac{dX_i^2}{U^2} + R^2 \frac{dU^2}{U^2} \quad . \quad (2.2)$$

In the  $R \rightarrow 0$  limit the hypersurface (2.1) becomes the lightcone space, and the metric (2.2) reduces into the  $D-1$ -dimensional conformally flat metric with conformal factor  $U^{-2}$ . The obtained space is  $(D - 1)$ -dimensional lightcone space described by the projective coordinates. After the limit the coordinate  $U$  becomes non-dynamical and the dimension of the space is reduced by one.  $U$  is the dilatation degree of freedom of the  $D$ -dimensional conformal symmetry.

It was generalized to supersymmetric case in [3]: The supersymmetric  $\text{AdS}_5 \times \text{S}^5$  space is described by a coset  $\text{GL}(4|4)/(\text{Sp}(4) \otimes \text{GL}(1))^2$  which is obtained by Wick rotations and introducing gauged degrees of freedom from a coset  $\text{PSU}(2,2|4)/\text{SO}(4,1) \otimes \text{SO}(5)$  [13]. After the projective lightcone limit the coset becomes  $\text{GL}(4|4)/\text{GL}(2|2)^2+$  and the obtained space is 4-dimensional flat space with  $N = 4$  superconformal symmetry which is 4-dimensional projective lightcone space.

We start with a simple 2-dimensional AdS space. Its isometry group is  $\text{SL}(2)$  and it is described by parameters of a coset  $\text{G}/\text{H} = \text{SL}(2)/\text{GL}(1)$ . For simpler treatment a coset  $\text{GL}(2)/\text{GL}(1)^2$  is used by introducing one more coordinate with one constraint. A  $\text{GL}(2)$  matrix is parametrized as

$$z = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \quad (2.3)$$

with real coordinates  $X, Y, u$  and  $v$ . Its inverse is

$$z^{-1} = \begin{pmatrix} 1 & -Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -X & 1 \end{pmatrix} \quad , \quad (2.4)$$

and the LI one form becomes

$$J_A{}^B = z^{-1} dz = \begin{pmatrix} j_u & j_Y \\ j_X & j_v \end{pmatrix}$$

$$= \begin{pmatrix} \frac{du}{u} - Y \frac{u}{v} dX & dY + \left( \frac{du}{u} - \frac{dv}{v} \right) Y - \frac{u}{v} dXY^2 \\ \frac{u}{v} dX & \frac{dv}{v} + \frac{u}{v} dXY \end{pmatrix} . \quad (2.5)$$

We choose the basis of Lie algebra of G and H as follows

$$\mathcal{G} = \mathfrak{gl}(2) = \{\tau_{+\rho}, \tau_{-\rho}, \tau_3, \mathbf{1}\} , \quad \mathcal{H} = \mathfrak{gl}(1)^2 = \{\tau_{+\rho}, \mathbf{1}\} \quad (2.6)$$

where  $\rho$  is a real parameter and

$$\tau_{\pm\rho} = \frac{\tau_{\pm} \pm \rho^2 \tau_{\mp}}{\rho} = \begin{pmatrix} 0 & 1/\rho \\ \pm\rho & 0 \end{pmatrix} , \quad \tau_{\pm} = \frac{\tau_1 \pm i\tau_2}{2} \quad (2.7)$$

$$[\tau_{+\rho}, \tau_{-\rho}] = -2\tau_3 , \quad [\tau_{\pm\rho}, \tau_3] = -2\tau_{\mp\rho} .$$

The basis  $\tau_M = \{\tau_{+\rho}, \tau_{-\rho}, \tau_3, \tau_0 = \mathbf{1}\}$  are normalized as

$$\left| (\tau_M)_A{}^B (\tau_N)_C{}^D \Omega^{AC} \Omega_{BD} \right| = 2\delta_{MN} \quad (2.8)$$

for  $\Omega_{AB} = \epsilon_{AB}$ . The LI one form is decomposed as

$$J_A{}^B = J_M (\tau_M)_A{}^B$$

$$J_{\pm\rho} = \frac{1}{2} \left( \rho j_Y \pm \frac{j_X}{\rho} \right) , \quad J_3 = \frac{1}{2} (j_u - j_v) , \quad J_0 = \frac{1}{2} (j_u + j_v) . \quad (2.9)$$

A coset element of G/H of the LI one form is written as

$$\langle J \rangle_A{}^B = J_{-\rho} (\tau_{-\rho})_A{}^B + J_3 (\tau_3)_A{}^B . \quad (2.10)$$

Under the local H-transformation  $z \rightarrow zh$  with  $h \in \mathcal{H}$

$$\langle J \rangle \rightarrow h^{-1} \langle J \rangle h , \quad (2.11)$$

the bilinear of the coset part current is invariant

$$\langle J \rangle_A{}^B \langle J \rangle_C{}^D \Omega^{AC} \Omega_{BD} = \langle J \rangle_A{}^B \langle J \rangle_C{}^D \left( h^{-1}{}^T \Omega h^{-1} \right)^{AC} \left( h \Omega h^T \right)_{BD} \quad (2.12)$$

from  $m \Omega m^T = (\det m) \Omega$  for an arbitrary GL(2) matrix  $m$ . The spacetime metric is

$$\begin{aligned} ds^2 &= \rho^2 \langle J \rangle_A{}^B \langle J \rangle_C{}^D \Omega^{AC} \Omega_{BD} \\ &= 2\rho^2 \left( -J_{-\rho}^2 + J_3^2 \right) \\ &= \frac{1}{2} \left\{ -\left( \rho^2 j_Y - j_X \right)^2 + \rho^2 (j_u - j_v)^2 \right\} \end{aligned} \quad (2.13)$$

In the  $\rho \rightarrow 0$  limit the metric (2.13) reduces into

$$ds^2 = -\frac{1}{2} j_X^2 = \frac{dX^2}{U^2} \quad (2.14)$$

with  $U = v/u \neq 0$ . This is nothing but the plc metric, (2.2) in  $R \rightarrow 0$  limit. The global  $G=GL(2)$  transformation,  $z \rightarrow z' = gz$  with  $g \in G$  is symmetry of the space (2.14)

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad X' = \frac{c + dX}{a + bX} \quad , \quad U' = \frac{(ad - bc)U}{(a + bX)^2} \quad \Rightarrow \quad \frac{dX'}{U'} = \frac{dX}{U} \quad . \quad (2.15)$$

In order to trace the local H symmetry relating to the local gauge symmetry in the limit we analyze the system canonically. We begin by the Lagrangian for a particle in the coset space (2.13)

$$L = \frac{1}{2} \left[ - \left\{ -\frac{1 + \rho^2 Y^2}{U} \dot{X} + \rho^2 \dot{Y} - \rho^2 \frac{Y \dot{U}}{U} \right\}^2 + \rho^2 \left( -\frac{\dot{U}}{U} - 2 \frac{Y}{U} \dot{X} \right)^2 \right] \quad . \quad (2.16)$$

Only  $U$  appears in  $L$  resulting  $GL(2)/GL(1)$  at this stage. Conjugate momenta are

$$\begin{cases} p = \frac{\partial L}{\partial \dot{X}} = \frac{2\rho}{U} (1 + \rho^2 Y^2) J_{-\rho} - \frac{4\rho^2 Y}{U} J_3 \\ \bar{p} = \frac{\partial L}{\partial \dot{Y}} = -2\rho^3 J_{-\rho} \\ \pi = \frac{\partial L}{\partial \dot{U}} = \frac{2\rho^2}{U} (\rho Y J_{-\rho} - J_3) \end{cases} \quad .$$

The coset part currents are rewritten as

$$J_{-\rho} = -\frac{\bar{p}}{2\rho^3} \quad , \quad J_3 = -\frac{1}{2\rho^2} (Y\bar{p} + U\pi) \quad . \quad (2.17)$$

The lack of the kinetic term for  $J_{+\rho}$  gives rise to a primary constraint

$$\phi \equiv Up - 2UY\pi + \left( \frac{1}{\rho^2} - Y^2 \right) \bar{p} = 0 \quad . \quad (2.18)$$

This will be identified with the local H-symmetry generator corresponding to  $\tau_{+\rho}$ . The generators of the local “right” action are given by

$$\phi_M = p\delta_M X + \bar{p}\delta_M Y + \pi\delta_M U \quad , \quad z \rightarrow ze^{\epsilon^M \tau_M} = z + \delta_M z \quad , \quad (2.19)$$

and they are

$$\begin{cases} \phi_{\pm\rho} = \rho \left\{ Up \mp 2UY\pi + \left( \frac{1}{\rho^2} \mp Y^2 \right) \bar{p} \right\} \\ \phi_3 = -2(Y\bar{p} + U\pi) \end{cases} \quad . \quad (2.20)$$

The constraint (2.18) is the local H-transformation generator corresponding to  $\tau_{+\rho}$ ,  $\phi = \phi_{+\rho}/\rho$ .

The Hamiltonian is obtained as

$$\begin{aligned} H &= p\dot{X} + \bar{p}\dot{Y} + \pi\dot{U} - L \\ &= \frac{1}{2} \left( \frac{U p}{1 + \rho Y} - \frac{\pi}{\rho U} \right) \left( \frac{U p}{-1 + \rho Y} - \frac{\pi}{\rho U} \right) . \end{aligned} \quad (2.21)$$

The local  $\tau_{+\rho} \in \mathcal{H}$  transformation is the gauge symmetry generator guaranteed by first classness,  $\dot{\phi} = \{\phi, H\} \approx 0$ . Using this gauge degree of freedom we fix the gauge,  $Y = 0$  with  $\{Y, \phi\} \neq 0$ , in such a way that the gauge fixed Hamiltonian becomes a simple form

$$H_{g.f.} = \frac{1}{2} \left( -U^2 p^2 + \frac{\pi^2}{\rho^2 U^2} \right) . \quad (2.22)$$

The gauge fixed Lagrangian becomes

$$L_{g.f.} = p\dot{X} + \pi\dot{U} - H_{g.f.} = \frac{1}{2} \left( -\frac{\dot{X}^2}{U^2} + \rho^2 \frac{\dot{U}^2}{U^2} \right) . \quad (2.23)$$

In the limit  $\rho \rightarrow 0$  the 2-dimensional AdS space (2.16) reduces into the 1-dimensional plc space

$$\xrightarrow{\rho \rightarrow 0} L_{\text{plc}} = -\frac{1}{2} \frac{\dot{X}^2}{U^2} . \quad (2.24)$$

Now  $U$  is nondynamical, so we face to have a new constraint  $\pi = 0$  originated to the local  $\tau_3$  transformation. The  $\phi_{+\rho}$  transformation constraint in (2.20) reduce into the  $\bar{p} = 0$  constraint in  $\rho \rightarrow 0$  limit. Using this constraint the  $\phi_3$  transformation generator reduces into  $\pi = 0$ . The consistency condition requires

$$\dot{\pi} = \{\pi, H_{\text{plc}}\} = U p^2 = 0 \quad , \quad H_{\text{plc}} = -\frac{1}{2} U^2 p^2 \quad , \quad (2.25)$$

so the invariance of the action  $\delta \int L_{\text{plc}} = 0$  is given by

$$\delta X = \xi \dot{x} \quad , \quad \delta U = \xi \dot{U} + \frac{1}{2} \dot{\xi} U \quad . \quad (2.26)$$

The gauge symmetry originated  $\tau_3$  transformation becomes the 1-dimensional general coordinate transformation in the plc limit. The plc system has local gauge invariance. We regard the local symmetry generated by  $\bar{p} = 0$  and  $\pi = 0$  as those from the stability group of a coset,  $\mathbb{H}$ , then

$$\mathcal{G} = \text{gl}(2) = \{\sqrt{2}\tau_+, \sqrt{2}\tau_-, \tau_3, \mathbf{1}\} \quad , \quad \mathcal{H} = \text{gl}(1)^2_+ = \{\tau_3, \mathbf{1}, \sqrt{2}\tau_+\} \quad . \quad (2.27)$$

This coset is called “half coset” which was introduced in [3]; the subgroup is triangle subgroup where diagonal parts are generated by  $\tau_3$  and  $\mathbf{1}$  and an upper-right part is generated by  $\tau_+$ . The coset is represented only by a lower-left part generated by  $\tau_-$ . The factor  $\sqrt{2}$  comes from the definition of  $\tau_{\pm}$  in (2.7) and it is normalized as (2.8). The coset

parameter  $X$  corresponding to  $\tau_-$  is a dynamical coordinate of the 1-dimensional space and is transformed under the global 1-dimensional conformal transformation,  $G=GL(2)$ , as (2.15). Although  $U$  corresponding to  $\tau_3$  is nondynamical in the  $\rho \rightarrow 0$  limit, it is indispensable for the  $G=GL(2)$  invariance (2.15).

Let us compare the plc limit with the IW contraction. For a Lie group  $G$  its Lie algebra is denoted by  $\mathcal{G} = \{T_M\}$ . The linear transformation of the generators  $T'_M = V_M^N T_N$  does not change the group if the transformation is nonsingular,  $\det V_M^N \neq 0$ . For the IW contraction the singular transformation is considered in the  $\rho \rightarrow 0$  limit as  $\det V_M^N(\rho) = \rho^\nu$  where  $\nu$  is the number of the contracted dimension [14]. Then new group  $G'$  generated by  $\{T'_M\}$  is different from original group  $G$ . On the other hand for the plc limit the linear transformation is nonsingular even in the  $\rho \rightarrow 0$  limit

$$V_M^N = \begin{pmatrix} \frac{1+\rho^2}{2\rho} & \frac{1-\rho^2}{2\rho} & 0 \\ \frac{1-\rho^2}{2\rho} & \frac{1+\rho^2}{2\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det V_M^N = 1 \quad (2.28)$$

where  $\{T_M\} = \{\tau_1, i\tau_2, \tau_3\}$  and  $\{T'_M\} = \{\tau_{+\rho}, \tau_{-\rho}, \tau_3\}$ . So the plc limit does not change the group  $G$ . However the Lie algebra of  $H$  for a coset  $G/H$  becomes nilpotent in the  $\rho \rightarrow 0$  limit. The coset  $G/H$  is a symmetric space for nonzero  $\rho$ , but is not so in the  $\rho \rightarrow 0$  limit breaking the gauge invariance of the action. In order to recover the gauge invariance of the action the kinetic term for the diagonal part ( $\tau_3$  component) is contracted to “0” and the corresponding degree of freedom is gauged. As a result the subgroup  $H$  is changed to new  $H'$  which is larger than  $H$ . Therefore the number of the coset parameter for  $G/H'$  is smaller than the one for  $G/H$ . This subgroup  $H'$  is sum of the diagonal part,  $H'_0$ , and the nilpotent part. Since the number of coset parameters of  $G/H'$  is one half of the one for  $G/H'_0$  which is a symmetric space, we denote it as the half coset  $G/H'_0+$ .

## 2.2 Generalization of projective lightcone limit

We generalize the above projective lightcone limit to “projective space limit” of a coset  $G/H$ . A coset element of  $G/H \ni z$  is transformed as  $z \rightarrow gzh$  with  $g \in G$ ,  $h \in H$ .

1. If a coset element is parametrized as

$$z = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \quad (2.29)$$

where  $u$  and  $v$  are square matrices and  $X$  and  $Y$  are rectangular matrices, then  $X$  is projective coordinate which is transformed as

$$\begin{aligned} z &\rightarrow gz, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ X &\rightarrow (c + dX)(a + bX)^{-1}. \end{aligned} \quad (2.30)$$

with the following transformation

$$\begin{aligned} u &\rightarrow (a + bX)u \\ v &\rightarrow dv - (c + dX)(a + bX)^{-1}bv \\ Y &\rightarrow Y + u^{-1}(a + bX)^{-1}bv \quad . \end{aligned} \quad (2.31)$$

The projective coordinate  $X$  represents the global group  $G$  by the linear fractional transformation.

2. There exists a projective space on which the global  $G$  symmetry is represented by the projective coordinate  $X$ . The metric of the projective space is given by  $ds^2 = J_X^2$  up to normalization, where  $J_X$  is the lower-left part of the LI one form  $z^{-1}dz$  as in the case of (2.14). This is obtained by the projective space limit of the metric constructed in a local  $H$ -invariant way in terms of maximal number of coordinates (2.13). At first rescale  $z$  as

$$z \rightarrow z \begin{pmatrix} 1/\sqrt{\rho} & 0 \\ 0 & \sqrt{\rho} \end{pmatrix} \quad , \quad (2.32)$$

then the LI one form,  $J = z^{-1}dz$ , is scaled as

$$J \rightarrow \begin{pmatrix} J_u & \rho J_Y \\ J_X/\rho & J_v \end{pmatrix} \quad . \quad (2.33)$$

Taking  $\rho \rightarrow 0$  limit in the metric which is written as bilinear form of the LI currents, only the  $J_X$  component is survived as in (2.14).

### 3 Complex projective space limit

We apply the above procedure to  $G=\text{SU}(n)$  case. At first we examine  $\text{SU}(2)$  as the simplest case. We present concrete correspondence between  $\text{SU}(2)$  coset element and coordinate system of the sphere  $S^3$ . Then the generalized projective space limit is taken resulting  $S^2$  or  $\text{CP}^1$ . Next we examine  $\text{SU}(n)$  case.

#### 3.1 $\text{SU}(2)$ : $S^3$ to $S^2$

A 3-dimensional sphere is described by three parameters of  $\text{SU}(2)$ . Instead we use four coordinates and one constraint as coset parameters of  $\text{GL}(2)/\text{GL}(1)$  which is Wick rotated  $\text{U}(2)/\text{U}(1)$ . A  $\text{GL}(2)$  matrix is parametrized as same as (2.3)

$$z = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix} \quad (3.1)$$



and  $z$  and  $z^{-1}dz$  have the same form as (2.4) and (2.5). Then we go back to  $U(2)$  by imposing the unitarity condition on  $z$ ;  $z^\dagger z = \mathbf{1}$ . Its hermite conjugate is given by

$$z^\dagger = \begin{pmatrix} 1 & 0 \\ Y^* & 1 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & v^* \end{pmatrix} \begin{pmatrix} 1 & X^* \\ 0 & 1 \end{pmatrix} . \quad (3.2)$$

The unitarity gives the following relations

$$|u|^2 = \frac{1}{1 + |X|^2} , \quad |v|^2 = 1 + |X|^2 , \quad Y = -u^* v X^* \quad (3.3)$$

with  $|u|^2 = u^* u$  and so on. It leads to  $|X|^2 = |Y|^2$ , so  $Y = 0$  gauge can not be chosen in this case. The LI one form satisfies the anti-hermiticity relation,  $(z^{-1}dz)^\dagger = -z^{-1}dz$ .

The 3-dimensional sphere is parametrized by  $SU(2)$  element  $z$  which satisfies

$$\sum_{A=0,1} z^\dagger_0{}^A z_A^0 = \sum_{A=0,1} z_A^0{}^* z_B^0 \delta^{AB} = 1 \quad (3.4)$$

for complex coordinates  $z$ . We identify  $z$  with (3.1), and write down a metric for  $S^3$  as

$$ds^2 = \sum_{A,B=0,1} \left( J_A^0 \right)^* J_B^0 \delta^{AB} \delta_{00} . \quad (3.5)$$

The coset element (3.1) is transformed as  $z \rightarrow gz$  with  $U(2) \ni g$ ,  $z$  and the LI one forms are manifestly invariant under it. Under the local  $U(1)$  transformation  $z \rightarrow zh$  with  $h = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}$ , the LI one form is transformed as

$$J_A^B \rightarrow \left( h^{-1} J h \right)_A^B + \begin{pmatrix} 0 & 0 \\ 0 & id\beta \end{pmatrix} . \quad (3.6)$$

The metric (3.5) is invariant under the above  $U(1)$  transformation from the unitarity condition of  $h$ ,

$$\left( h^{-1*} \right)_A^C \left( h^{-1} \right)_B^D \delta^{AB} = \delta^{CD} , \quad (h^*)_0^0 (h)_0^0 \delta_{00} = \delta_{00} . \quad (3.7)$$

So the metric of the 3-dimensional sphere (3.5) has both global  $U(2)$  symmetry and the local  $U(1)$  symmetry. The first term of the metric (3.5) becomes

$$\begin{aligned} \left( J_0^0 \right)^* J_0^0 &= \left( \frac{du}{u} - Y \frac{u}{v} dX \right)^* \left( \frac{du}{u} - Y \frac{u}{v} dX \right) \\ &= \left( d\phi + \frac{i}{2} \frac{X d\bar{X} - dX \bar{X}}{1 + |X|^2} \right)^2 \end{aligned} \quad (3.8)$$

where we use new variables determined from (3.3)  $u = e^{i\phi}/\sqrt{1 + |X|^2}$ . The second term of the metric (3.5) becomes

$$\left( J_1^0 \right)^* J_1^0 = \left( \frac{u}{v} dX \right)^* \left( \frac{u}{v} dX \right) = \frac{|dX|^2}{(1 + |X|^2)^2} . \quad (3.9)$$

The metric (3.9) is nothing but the metric of a 2-dimensional sphere.

Total metric (3.5) for a 3-dimensional sphere is given as

$$\begin{aligned} ds^2 &= \left( d\phi + \frac{i}{2} \frac{X d\bar{X} - dX \bar{X}}{1 + |X|^2} \right)^2 + \frac{|dX|^2}{(1 + |X|^2)^2} \\ &= \frac{1}{1 + |\tilde{X}|^2} \left( d\phi^2 + |d\tilde{X}|^2 \right) - \frac{1}{4} \frac{1}{(1 + |\tilde{X}|^2)^2} d(|\tilde{X}|)^2 \end{aligned} \quad (3.10)$$

with  $\tilde{X} = e^{i\phi} X$ . Changing variables as  $|\tilde{X}|^2 = r^2$ ,  $|d\tilde{X}|^2 = dr^2 + r^2 d\chi^2$  it leads to

$$ds^2 = \frac{dr^2}{(1 + r^2)^2} + \frac{1}{1 + r^2} d\phi^2 + \frac{r^2}{1 + r^2} d\chi^2 \quad . \quad (3.11)$$

Further changing  $r = \tan \theta$  leads to

$$ds^2 = d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\chi^2 \quad (3.12)$$

with  $0 \leq \theta \leq \pi/2$ ,  $-\pi \leq \phi \leq \pi$ ,  $0 \leq \chi \leq \pi$ . This metric represents a 3-dimensional sphere which is embedded as

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 &= 1 \\ x &= \cos \theta \cos \phi, \quad y = \cos \theta \sin \phi, \quad z = \sin \theta \cos \chi, \quad w = \sin \theta \sin \chi \end{aligned} \quad (3.13)$$

The radius of the sphere  $R$  is introduced by replacing  $X$  by  $X/R$  and  $ds^2$  by  $R^2 ds^2$  as

$$ds^2 = R^2 \left( d\phi + \frac{i}{2} \frac{X d\bar{X} - dX \bar{X}}{R^2 + |X|^2} \right)^2 + \frac{R^4 |dX|^2}{(R^2 + |X|^2)^2} \quad (3.14)$$

giving the scalar curvature  $4/R^2$ . In the large radius limit,  $R \rightarrow \infty$  the curvature becomes zero, and the second term of (3.14) reduces into the 2-dimensional flat space while the first term becomes one more flat direction with the coordinate  $-\infty \leq R\phi \leq \infty$ .

Now we perform the complex projective space limit by following the subsection 2.2.

1. As in the equation (2.30) the  $X$  is complex projective coordinate which is transformed under the global  $U(2) \ni g$ ,  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  as

$$X \rightarrow X' = \frac{c + dX}{a + bX} \quad . \quad (3.15)$$

2. As in the equation (2.33) through the rescaling the coordinates the LI one forms are scaled as

$$J_A^B \rightarrow \begin{pmatrix} J_0^0 & \rho J_0^1 \\ \frac{1}{\rho} J_1^0 & J_1^1 \end{pmatrix} \quad . \quad (3.16)$$

Then the metric in  $\rho \rightarrow 0$  limit becomes

$$\begin{aligned} ds^2 &= \rho^2 R^2 \left( d\phi + \frac{i}{2} \frac{X d\bar{X} - dX \bar{X}}{R^2 + |X|^2} \right)^2 + \frac{R^4 |dX|^2}{(R^2 + |X|^2)^2} \\ &\xrightarrow{\rho \rightarrow 0} \frac{R^4 |dX|^2}{(R^2 + |X|^2)^2} \end{aligned} \quad (3.17)$$

which is the 2-dimensional sphere metric in terms of the complex coordinate. It is well known that a 2-dimensional sphere is described by Riemannian surface  $\mathbb{CP}^1$ ; the 2-dimensional plane or 1-dimensional complex plane projected stereographically of the sphere plus a point at infinity. The resultant coset is  $U(2)/U(1)^2$ , since additional constraint  $\pi_\phi = 0$  corresponds to additional  $U(1)$  in the subgroup.

### 3.2 $SU(n)$ : $S^{2n-1}$ to $\mathbb{CP}^{n-1}$

Let us consider  $S^{2n-1}$  space by taking  $SU(n)$  symmetry. Analogous to the previous section we use  $GL(n)/GL(n-1)$  instead of  $SU(n)/SU(n-1)$  by Wick rotation and introducing gauge coordinates. The parametrization of  $GL(n)$ ,  $z$ , is given by as

$$z_M^A = \begin{pmatrix} z_0^0 & z_0^j \\ z_i^0 & z_i^j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ X & \mathbf{1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \mathbf{v} \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & \mathbf{1} \end{pmatrix} \quad , \quad i, j = 1, \dots, n-1 \quad . \quad (3.18)$$

Its inverse is

$$z^{-1} = \begin{pmatrix} 1 & -Y \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & \mathbf{v}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -X & \mathbf{1} \end{pmatrix} \quad , \quad (3.19)$$

and the Left invariant one form becomes

$$z^{-1} dz = \begin{pmatrix} \frac{du}{u} - Y \mathbf{v}^{-1} dX u & dY + \frac{du}{u} Y - Y \mathbf{v}^{-1} d\mathbf{v} - Y \mathbf{v}^{-1} dX u Y \\ \mathbf{v}^{-1} dX u & \mathbf{v}^{-1} d\mathbf{v} + \mathbf{v}^{-1} dX u Y \end{pmatrix} . \quad (3.20)$$

Then we go back to  $U(n)$  by imposing the unitarity condition on  $z$ ,  $z^\dagger z = \mathbf{1}$  where its hermite conjugate is given by

$$z^\dagger = \begin{pmatrix} 1 & 0 \\ Y^\dagger & \mathbf{1} \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & \mathbf{v}^\dagger \end{pmatrix} \begin{pmatrix} 1 & X^\dagger \\ 0 & \mathbf{1} \end{pmatrix} \quad . \quad (3.21)$$

The unitary condition gives the following relations

$$\begin{aligned} |u|^2 &= \frac{1}{1 + |X|^2} \quad , \quad Y = -u^* X^\dagger \mathbf{v} \\ (\mathbf{v} \mathbf{v}^\dagger)_i^j &= \delta_i^j + X_i X^{\dagger j} = \Upsilon_i^j \quad , \quad \Upsilon^{-1}_i{}^j = \delta_i^j - \frac{X_i X^{\dagger j}}{1 + |X|^2} \end{aligned} \quad (3.22)$$

satisfying  $|X|^2 = |Y|^2$  with  $|X|^2 = \sum_{i=1}^{n-1} (X_i)^* X_i$ .

A  $(2n-1)$ -dimensional sphere is parametrized by  $SU(n)/SU(n-1)$  parameters as

$$\sum_{A=0,1,\dots,n-1} z_0^\dagger{}^A z_A^0 = \sum_{A=0,1,\dots,n-1} z_A^{0*} z_B^0 \delta^{AB} = 1 \quad . \quad (3.23)$$

We identify  $z$  with (3.18), and write down a metric of  $S^{2n-1}$  as

$$ds^2 = \sum_{A,B=0}^{n-1} (J_A^0)^* J_B^0 \delta^{AB} \delta_{00} \quad . \quad (3.24)$$

This is invariant under the local H transformation: Under a H transformation,  $U(n-1) \ni h$ ,  $h = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$  with  $\beta^\dagger \beta = \mathbf{1}$  the LI one forms are transformed as

$$J_A^B \rightarrow (h^{-1} J h)_A^B + \begin{pmatrix} 0 & 0 \\ 0 & \beta^\dagger d\beta \end{pmatrix} \quad . \quad (3.25)$$

The metric (3.24) is invariant under  $h$  from

$$\left( (h^{-1})^* \right)_A^C \left( h^{-1} \right)_B^D \delta^{AB} = \delta^{CD} \quad , \quad (h^*)_0^0 (h)_0^0 \delta_{00} = \delta_{00} \quad . \quad (3.26)$$

The first term of the metric (3.24) becomes

$$\begin{aligned} (J_0^0)^* J_0^0 &= \left[ \frac{du}{u} - Y \mathbf{v}^{-1} dX u \right]^* \left[ \frac{du}{u} - Y \mathbf{v}^{-1} dX u \right] \\ &= (d\phi + A)^2 \\ A &= \frac{i \sum_{i=1}^{n-1} (X_i d\bar{X}^i - dX_i \bar{X}^i)}{2 \frac{1 + |X|^2}{1 + |X|^2}} \end{aligned} \quad (3.27)$$

where we use  $u = e^{i\phi} / \sqrt{1 + |X|^2}$  from (3.22). The rest terms become

$$\begin{aligned} \sum_{i=1}^{n-1} (J_i^0)^* J_i^0 &= \sum_{i=1}^{n-1} [\mathbf{v}^{-1} dX u]_i^* [\mathbf{v}^{-1} dX u]_i \\ &= \sum_{i,k=1}^{n-1} \frac{d\bar{X}^i}{1 + |X|^2} \left( \mathbf{1}_i^k - \frac{X_i \bar{X}^k}{1 + |X|^2} \right) dX_k \end{aligned} \quad (3.28)$$

which is the Fubini-Study metric for a  $(n-1)$ -dimensional complex projective space. The total metric for a  $(2n-1)$ -dimensional sphere is given by

$$\begin{aligned} ds^2 &= (d\phi + A)^2 + \sum_{i,k=1}^{n-1} \frac{d\bar{X}^i}{1 + |X|^2} \left( \mathbf{1}_i^k - \frac{X_i \bar{X}^k}{1 + |X|^2} \right) dX_k \\ &= \frac{d\phi^2 + \sum_{i=1}^{n-1} d\tilde{X}^i d\tilde{X}_i}{1 + |\tilde{X}|^2} - \left( \frac{1}{2} \frac{d \sum_{i=1}^{n-1} \tilde{X}^i \tilde{X}_i}{1 + |\tilde{X}|^2} \right)^2 \end{aligned} \quad (3.29)$$

with  $\tilde{X} = e^{i\phi} X$ . Changing variables as

$$|\tilde{X}|^2 = r^2, \quad |d\tilde{X}|^2 = dr^2 + r^2 d\Omega_{(2n-3)}^2 \quad (3.30)$$

leads to

$$ds^2 = \frac{dr^2}{(1+r^2)^2} + \frac{1}{1+r^2} d\phi^2 + \frac{r^2}{1+r^2} d\Omega_{(2n-3)}^2. \quad (3.31)$$

Further rewriting as  $r = \tan \theta$

$$ds^2 = d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_{(2n-3)}^2. \quad (3.32)$$

This metric gives constant positive curvature describing the  $(2n-1)$ -dimensional sphere. The radius of the sphere  $R$  is inserted back as

$$ds^2 = R^2 (d\phi + A)^2 + \sum_{i,k=1}^{n-1} \frac{R^2 d\bar{X}^i}{R^2 + |X|^2} \left( \mathbf{1}_i^k - \frac{X_i \bar{X}^k}{R^2 + |X|^2} \right) dX_k$$

$$A = \frac{i}{2} \frac{\sum_{i=1}^{n-1} (X_i d\bar{X}^i - dX_i \bar{X}^i)}{R^2 + |X|^2} \quad (3.33)$$

which reduces into the  $(2n-1)$ -dimensional flat space metric in  $R \rightarrow 0$  limit where the second term in (3.33) becomes  $(2n-2)$ -dimensional flat metric and the first term becomes one more coordinate  $-\infty \leq R\phi \leq \infty$ .

Now let us perform the limiting procedure analogously to the previous subsection.

1. As in the equation (2.30) the  $X_i = z_i^0/z_0^0$  are projective coordinates which are transformed under the global  $U(n)$  transformation as

$$X_i \rightarrow \frac{c_i + \sum_{k=1}^{n-1} \mathbf{d}_i^k X_k}{a + \sum_{j=1}^{n-1} b^j X_j}, \quad g = \begin{pmatrix} a & b^j \\ c_i & \mathbf{d}_i^j \end{pmatrix} \in U(n). \quad (3.34)$$

2. As in the equation (2.33) through the rescaling the coordinates the LI one forms are rescaled as

$$J_A^B \rightarrow \begin{pmatrix} J_0^0 & \rho J_0^j \\ \frac{1}{\rho} J_i^0 & J_i^j \end{pmatrix}. \quad (3.35)$$

Now let us take the  $\rho \rightarrow 0$  limit in the metric

$$ds^2 = \rho^2 R^2 (d\phi + A)^2 + \sum_{i,k=1}^{n-1} \frac{R^2 d\bar{X}^i}{R^2 + |X|^2} \left( \mathbf{1}_i^k - \frac{X_i \bar{X}^k}{R^2 + |X|^2} \right) dX_k$$

$$\begin{aligned}
& \xrightarrow{\rho \rightarrow 0} \sum_{i,j=1}^{n-1} \frac{R^2 d\bar{X}^i}{R^2 + |X|^2} \left( \mathbf{1}_i^j - \frac{X_i \bar{X}^j}{R^2 + |X|^2} \right) dX_j \\
A &= \frac{i}{2} \frac{\sum_{i=1}^{n-1} (X_i d\bar{X}^i - dX_i \bar{X}^i)}{R^2 + |X|^2}
\end{aligned} \tag{3.36}$$

with  $\bar{X}^i = X_i^*$ . Disappearance of the kinetic term for  $\phi$  leads to a new constraint  $\pi_\phi = 0$  corresponding to additional U(1) in the subgroup:  $G/H$  with  $G=U(n)$  and  $H=U(n-1) \otimes U(1)$ . The obtained metric (3.36) is the Fubini-Study metric for the  $(n-1)$ -dimensional complex projective space,  $CP^{n-1}$ . It is a constant positive curvature space but it is not expressed as the hypersurface in the Euclidean space. The complex projective space metric is given in terms of the Kähler expression

$$g_{i\bar{j}} = \frac{1}{1 + |X|^2} \left( \mathbf{1}_i^j - \frac{X_i \bar{X}^j}{1 + |X|^2} \right) = \frac{\partial}{\partial \bar{X}^i} \frac{\partial}{\partial X_j} K \tag{3.37}$$

with the Kähler potential

$$K = \ln(1 + |X|^2) = -\ln |z_0^0|^2 = -\ln |u|^2, \tag{3.38}$$

from the fact that  $\sum_{A=0}^{n-1} |z_A^0|^2 = 1 = \left( 1 + \sum_{A=1}^{n-1} |X_A^0|^2 \right) \cdot |z_0^0|^2 = (1 + |X|^2) \cdot |z_0^0|^2$ .

## 4 Conclusion and discussion

We have discussed the projective lightcone limit of an AdS space with clarifying local symmetries in each step of the limit. In the plc limit the kinetic term corresponding to the box diagonal element is contracted to zero resulting an additional local gauge symmetry. This is regarded as the change of the subgroup  $H$  into an upper triangle subgroup. The coset parameters are reduced into lower triangle matrix elements excluding the box diagonal part, and the number of spacetime coordinate is reduced by one. Although the box diagonal element becomes nondynamical, it is indispensable for realizing the global symmetry  $G$ .

We generalize this limit from a sphere to a complex projective space. Both spaces have  $U(n)$  symmetry. A  $(2n-1)$ -dimensional sphere is described by a coset  $G/H=U(n)/U(n-1)$ , while a  $(n-1)$ -dimensional complex projective space is described by  $G/H=U(n)/U(n-1) \otimes U(1)$ . This projective space limit corresponds to the Hopf reduction, where our method is a procedure to relate these spaces as a kind of group contraction preserving group symmetries of projective coordinates manifestly. The projective space limit  $S^3$  to  $S^2$  ( $CP^1$ ) is similar to the gauged nonlinear sigma model discussed in the subsections 4(C) and 4(D) of the third reference in [5] but different coordinates are used. Extension to  $U(n)$  case is straightforward for the generalized plc case. The generalized plc uses a  $U(n)$  matrix as a coordinate, while the gauged nonlinear sigma model uses  $U(n)$  vector.

Auxiliary degrees of freedom of  $U(n)$  matrix, which are box diagonal parts, are essential to give the Fubini-Study metric (3.29) systematically through (3.22). Further applications will be possible to supersymmetric cases, noncompact spaces, noncommutative spaces and T-dual spaces.

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